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Spinors, Algebraic Geometry, and the Classification of Second-Order Symmetric Tensors in General Relativity.

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Two approaches to the problem of classifying second-order symmetric tensors in space-time given by Ludwig and Scanlon and by Penrose are discussed. Ludwig and Scanlon use both spinor and tensor algebra in their approach, whereas Penrose uses spinors and the properties of certain curves in complex projective 3-space. These approaches yield essentially identical classifications, and this paper points out the connections between them in detail and tabulates the results.

1. INTRODUCTION

Several authors have discussed the classification of second-order symmetric tensors in general relativity [Plebański, 1964; Ludwig and Scanlon, 1971; Collinson and Shaw, 1972; Penrose, 1972; Hall, 1976 (for a review see Hall, 1979); Cormack and Hall, 1979a, 1979b; Shaw, 1971; Sobczyk, 1980]. Two of these approaches, namely, those of Ludwig and Scanlon and of Penrose, give particularly detailed versions of the classification. It turns out that these two approaches are, apart from a few details (which will be considered), essentially the same and it is the purpose of the present paper to point out the similarities in detail, especially as one of the papers, that of Penrose, is written in the language of algebraic geometry, a language not always entirely familiar to relativists.

The remainder of this section will be devoted to general points of notation and a brief discussion of the spinor approach to the classification problem which will be useful later in the paper. In Sections 2 and 3 the work of Ludwig and Scanlon and of Penrose will be reviewed. In Section 4 the connections between these approaches will be given and the results tabulated.

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Throughout the paper the problem of classifying the Ricci tensor in a space-time M will be considered, although the same methods apply equally well to any symmetric second-order tensor in a Lorentzian manifold taken here to have signature $+2$. The notation will be the same as that used in the papers by Hall (1976, 1979) and Cormack and Hall (1979a, 1979b). Without loss in generality, attention will be focused on the trace-free Ricci tensor since as far as the present discussion is concerned, it possesses the same algebraic structure as the Ricci tensor. If $p \in M$, $T_p(M)$ denotes the tangent space to M at p and using component notation in some chart about p, the Ricci tensor R_{ab}, its trace-free part $\tilde{R}_{ab}$, the Riemann, Weyl, and metric tensors and the Ricci scalar are connected in the usual notation by the relations

$$R_{abcd} = C_{abcd} + E_{abcd} + \tfrac{1}{6} R g_{a[c} g_{d]b} \quad \text{(a)}$$

$$E_{abcd} = \tilde{R}_{a[c} g_{d]b} + \tilde{R}_{b[d} g_{c]a} \quad \text{(b)}$$

$$R_{ab} = R^{c}_{\ acb}, \qquad \tilde{R}_{ab} = R_{ab} - \tfrac{1}{4} R g_{ab}, \qquad R = R_{ab} g^{ab} \quad \text{(c)} \quad (1.1)$$

$$^{*}E_{abcd} = -E^{*}_{abcd}, \qquad E^{*}_{abcd} = -E^{*}_{cdab} \quad \text{(d)}$$

$$E^{c}_{acb} = \tilde{R}_{ab}, \qquad E^{c*}_{acb} = 0 \quad \text{(e)}$$

The tensor E_{abcd} is completely equivalent to the trace-free Ricci tensor and has the algebraic symmetries of the Riemann tensor. It has been used as the basis of a classification of the Ricci tensor (Cormack and Hall, 1979a). The tensors $\tilde{R}_{ab}$ and $\overset{+}{E}_{abcd}$ $(= E_{abcd} + iE^{*}_{abcd})$ have spinor equivalents (in the usual notation[1]) $\tilde{R}_{ab} \leftrightarrow 2\phi_{AB\dot{X}\dot{Y}}$ and $\overset{+}{E}_{abcd} \leftrightarrow 2\varepsilon_{AB}\varepsilon_{\dot{Y}\dot{Z}}\phi_{CD\dot{W}\dot{X}}$, where $\phi_{AB\dot{X}\dot{Y}}$ has the symmetry and Hermitian properties

$$\phi_{[AB]\dot{X}\dot{Y}} = \phi_{AB[\dot{X}\dot{Y}]} = 0, \qquad \phi_{AB\dot{X}\dot{Y}} = \bar{\phi}_{\dot{X}\dot{Y}AB} \quad (1.2)$$

If one is interested in a classification of the spinor $\phi_{AB\dot{X}\dot{Y}}$, then an obvious approach is to consider the existence of symmetric eigen 2-spinors ψ^{AB} satisfying the eigenvalue problem

$$\psi^{AB}\phi_{AB\dot{X}\dot{Y}} = \lambda \bar{\psi}_{\dot{X}\dot{Y}} \qquad (\lambda \in \mathbb{C}) \quad (1.3)$$

[1]For the spinor notation see Pirani (1965), except that here the signature is $+2$ and not -2. This is of nuisance value in the spinor formalism but allows easier comparison with results in vector notation. Capital Latin letters take the values 1 and 2 and ε_{AB} is the alternating symbol in two dimensions.

the classification then being given by the number of independent solutions of (1.3) and their type (null or nonnull). By recalling that the tensor equivalent of a symmetric 2-spinor is a complex self-dual bivector $\overset{+}{F}_{ab}(\leftrightarrow 2\varepsilon_{\dot{X}\dot{Y}}\psi_{AB})$ equation (1.3) is seen to be equivalent to the eigenvalue problem

$$\overset{+}{E}_{abcd}\overset{+}{F}{}^{cd}=4\lambda\overset{+}{F}_{ab} \tag{1.4}$$

This eigenvalue problem has been dealt with (Cormack and Hall, 1979a) and shown to yield a straightforward classification of the trace-free Ricci tensor. The representation of $\tilde{R}_{ab}$ using the fourth-order tensor $\overset{+}{E}_{abcd}$ enables many parallels with the Petrov classification of the Weyl tensor to be drawn.

An alternative approach is to consider the eigenvalue problem

$$\phi_{AB\dot{X}\dot{Y}}\chi^{B\dot{Y}}=\mu\chi_{A\dot{X}} \qquad (\mu\in\mathbb{R}) \tag{1.5}$$

where $\chi_{A\dot{X}}$ is a Hermitian spinor. This is of course equivalent to the standard eigenvalue problem $\tilde{R}_{ab}k^b=-2\mu k_a$, where k^a is a real vector. In fact if one considers the simplified problem of finding decomposable (null) solutions $\chi_{A\dot{X}}=\alpha_A\bar{\alpha}_{\dot{X}}$ of (1.5), where α_A is a 1-spinor, one is led to a straightforward classification of $\tilde{R}_{ab}$ according to its null eigenvectors (cf. Hall, 1976). The relationship between the coefficients in the various canonical forms for R_{ab}, $\tilde{R}_{ab}$, $\overset{+}{E}_{abcd}$, and $\phi_{AB\dot{X}\dot{Y}}$ is given in the Appendix.

This section will be closed by listing that, which is the equivalent for $\overset{+}{E}_{abcd}$ and $\phi_{AB\dot{X}\dot{Y}}$, of the Bel criteria for the Weyl tensor and its spinor equivalent given by Bel (1962) and Penrose (1960). The proofs are easily gathered from the canonical forms listed in the appendix (cf. Cormack and Hall, 1979a), and in the statements it is assumed that $\overset{+}{E}_{abcd}$ and $\phi_{AB\dot{X}\dot{Y}}$ are nonzero.

(i) There exists $l\in T_p(M)$, with l nonzero such that $l^a\overset{+}{E}_{abcd}=0\Leftrightarrow$there exists a nonzero 1-spinor α at p such that $\alpha^A\phi_{AB\dot{X}\dot{Y}}=0$.

(ii) There exists $l\in T_p(M)$ and a nonzero self-dual null bivector $\overset{+}{F}$ at p such that $l^a\overset{+}{E}_{abcd}=l_b\overset{+}{F}_{cd}\Leftrightarrow$there exists a 1-spinor α at p such that $\alpha^A\phi_{AB\dot{X}\dot{Y}}\neq 0$ and $\alpha^A\alpha^B\phi_{AB\dot{X}\dot{Y}}=\alpha^A\bar{\alpha}^{\dot{X}}\phi_{AB\dot{X}\dot{Y}}=0$.

(iii) There exists $l\in T_p(M)$ with l nonzero and null such that $l^al^c\overset{+}{E}_{abcd}=\lambda l_bl_d(\lambda\in\mathbb{R})\Leftrightarrow$there exists a nonzero 1-spinor α at p such that $\alpha^A\bar{\alpha}^{\dot{X}}\phi_{AB\dot{X}\dot{Y}}=\rho\alpha_B\bar{\alpha}_{\dot{Y}}(\rho\in\mathbb{R})$, or equivalently, $\alpha^A\alpha^B\phi_{AB\dot{X}\dot{Y}}=\sigma\bar{\alpha}_{\dot{X}}\bar{\alpha}_{\dot{Y}}(\sigma\in\mathbb{C})$.

(iv) There exists $l\in T_p(M)$ with l nonzero and null such that $l_{[e}\overset{+}{E}_{a]bc[d}l_{f]}l^bl^c=0\Leftrightarrow$there exists a nonzero 1-spinor α at p such that $\phi_{AB\dot{X}\dot{Y}}\alpha^A\alpha^B\bar{\alpha}^{\dot{X}}\bar{\alpha}^{\dot{Y}}=0$.

In (i) the vector l and the spinor α are unique up to a real and a complex factor, respectively, and l is necessarily null being the vector equivalent of the spinor $\alpha_A\bar{\alpha}_{\dot{X}}$. The statement in (i) is equivalent to the Ricci tensor having Segré type $\{(2,1,1)\}$ with l generating the (unique) null Ricci eigendirection. In (ii) one again has that l and α are unique up their respective multiplicative factors and that l is necessarily null being the vector equivalent of $\alpha_A\bar{\alpha}_{\dot{X}}$. Also l generates the (unique) principal null direction of $\overset{+}{F}_{ab}$. The statement in (ii) is equivalent to the Ricci tensor having Segré type $\{(3,1)\}$ with l generating the (unique) null Ricci eigendirection. The statement in (iii) is equivalent to l generating a null Ricci eigendirection and this direction is not necessarily unique. The statement (iv) is equivalent to the statement that l is nonzero, null, and satisfies $R_{ab}l^a l^b = 0$. Again l is not necessarily unique.

2. THE LUDWIG–SCANLON CLASSIFICATION

The approach of Ludwig and Scanlon is based on the fact that the components of the trace-free Ricci tensor at $p \in M$ can always be written in the form (Ludwig and Scanlon, 1971)

$$2\tilde{R}_{ab} = r_{(a}s_{b)} + \bar{r}_{(a}\bar{s}_{b)} - \tfrac{1}{4}(r_c s^c + \bar{r}_c\bar{s}^c)g_{ab} \tag{2.1}$$

where r and s are complex vectors at p. They then classify $\tilde{R}_{ab}$ ($\neq 0$) according to the following scheme.

Type A. This occurs when r and s are real and proportional and gives two immediate subtypes according to the sign of the factor of proportionality.

Type B. This occurs when r and s are real but not proportional.

Type C. This occurs when r and s are complex vectors (not complex multiples of real vectors) where r is nonnull and where $s = \pm\bar{r}$. Again two subtypes occur according to the sign in the last equation.

Type D. This type comprises all the cases not covered in A, B or C.

For the first three of the above types one may write the trace-free Ricci tensor as

$$\begin{aligned} A\pm:&\quad \tilde{R}_{ab} = \pm\left(r_a r_b - \tfrac{1}{4}r_c r^c g_{ab}\right) &\text{(i)}\\ B:&\quad \tilde{R}_{ab} = r_{(a}s_{b)} - \tfrac{1}{4}r_c s^c g_{ab} &\text{(ii)}\\ C\pm:&\quad \tilde{R}_{ab} = \pm\left(r_{(a}\bar{r}_{b)} - \tfrac{1}{4}r_c\bar{r}^c g_{ab}\right) &\text{(iii)} \end{aligned} \tag{2.2}$$

In (2.2) (i) r is a real vector and is determined up to sign by a type-A trace-free Ricci tensor. Similarly, in (2.2) (ii), r and s are real and not parallel and determined up to the changes $r \to \kappa r$, $s \to \kappa^{-1}s$ ($\kappa \in \mathbb{R}$) and $r \to s$,

Table I.[a] A Detailed Comparison of the Ludwig–Scanlon, Segré, and Penrose Types

Ludwig–Scanlon type	Sign of relevant invariant r_ar^a	s_as^a	r_as^a	I	Segré type	Penrose type
$A_{1\pm}$	+				$\{(1,1,1)1\}$	$C_1^2\tau^\infty$
$A_{2\pm}$	−				$\{1(1,1,1)\}$	C^2
$A_{3\pm}$	0				$\{(2,1,1)\}$	X^2q
B_{1a}	0	0	−	+	$\{(1,1)(1,1)\}$	$XXii$
B_{1b}	0	0	+	+	$\{(1,1)(1,1)\}$	$XXii$
B_{2a}	+	0	0	0	$\{(3,1)\}$	C_1Xt
B_{2b}	+	0	−	+	$\{2(1,1)\}$	C_1Xi
B_{2c}	+	0	+	+	$\{2(1,1)\}$	C_1Xi
B_{3a}	+	+	0	−	$\{(1,1)1,1\}$	C_1C_1nn
B_{3b}	+	+	−	−	$\{(1,1)1,1\}$	C_1C_1nn
B_{3c}	+	+	+	−	$\{(1,1)1,1\}$	C_1C_1nn
B_{3d}	+	+	−	0	$\{(2,1)1\}$	$C_1C_1\tau$
B_{3e}	+	+	+	0	$\{(2,1)1\}$	$C_1C_1\tau$
B_{3f}	+	+	−	+	$\{1,1(1,1)\}$	C_1C_1
B_{3g}	+	+	+	+	$\{1,1(1,1)\}$	C_1C_1
B_{4a}	0	−	−	+	$\{2(1,1)\}$	CXi
B_{4b}	0	−	+	+	$\{2(1,1)\}$	CXi
B_{5a}	−	−	−	+	$\{1,1(1,1)\}$	CC
B_{5b}	−	−	+	+	$\{1,1(1,1)\}$	CC
B_{6a}	+	−	0	+	$\{z,\bar{z}(1,1)\}$	C_1C
B_{6b}	+	−	−	+	$\{z,\bar{z}(1,1)\}$	C_1C
B_{6c}	+	−	+	+	$\{z,\bar{z}(1,1)\}$	C_1C
$C_{1\pm}$				+	$\{1,1(1,1)\}$	B
$C_{2\pm}$				0	$\{(2,1)1\}$	$B\tau$
$C_{3\pm}$				−	$\{(1,1)1,1\}$	Bii
D_1	$\left\{\begin{array}{c}\rho_4 \gtrless \rho_1,\rho_2,\rho_3\\ \text{otherwise}\end{array}\right\}$				$\{1,1,1,1\}$	$\left\{\begin{array}{c}Q\\ Q_2\end{array}\right\}$
D_2					$\{z,\bar{z},1,1\}$	Q_1
D_3	$\left\{\begin{array}{l}\sigma_2 \gtrless \sigma_1 \gtrless \sigma_3\\ \sigma_1 \gtrless \sigma_2,\sigma_3,\ \sigma_1\lambda>0\\ \sigma_1 \gtrless \sigma_2,\sigma_3,\ \sigma_1\lambda<0\end{array}\right\}$				$\{2,1,1\}$	$\left\{\begin{array}{c}Q_1n\\ Q_1i\\ Qi\end{array}\right\}$
D_4					$\{3,1\}$	Q_1c

[a]The first five columns for types A, B, and C and the first and third columns for type D give the Ludwig–Scanlon classification. Here, $I=(r_as^a)^2-(r_ar^a)(s_bs^b)$ and is positive, negative, or zero according as the 2-space spanned by r and s is timelike, spacelike, or null. The eigenvalues occurring in the type-D case refer to equations (4.1) and (4.3). The sixth column (third column for type D) gives the Segré type, where the convention that all degeneracies are included inside parentheses and that in the diagonal case, the first digit corresponds to the timelike eigenvalue, is adopted. The final column gives the Penrose type.

$s \to r$. Again in (2.2) (iii) r is complex (but not a complex multiple of a real vector), nonnull and unique up to the changes $r \to e^{i\theta} r$ ($\theta \in \mathbb{R}$) and $r \to \bar{r}$, $\bar{r} \to r$. It is easily shown that the classes A, B, C, and D are exhaustive of nonzero trace-free Ricci tensors and disjoint, the nonnull condition on r in type C being necessary to keep the types B and C disjoint.

Ludwig and Scanlon subdivide the class A according to whether the vector r in (2.2) (i) is spacelike, timelike, or null. The class B is subdivided according to the signs or zeros of $r_a r^a$, $s_a s^a$, and $r_a s^a$ and the nature of the 2-space spanned by r and s (timelike, spacelike, or null). The class C is subdivided according to the nature of the 2-space spanned by the real and imaginary parts of r. This subclassification is of course independent of the form chosen for $\tilde{R}_{ab}$ within the above-mentioned ambiguities (except of course that in class B, $r_a r^a$ and $s_a s^a$ are interchanged if r and s are). The class D is subclassified according to Segré type. The classification scheme is given in full in Table I.

3. THE PENROSE CLASSIFICATION

One recalls the elegant spinor version of the Petrov classification given by Penrose (1960) (see also Pirani, 1965). Here, the relevant spinor is the completely symmetric four-index spinor equivalent of the complex self-dual Weyl tensor, ψ_{ABCD}. This spinor may be looked upon as determining a quartic equation $\psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0$ on the complex projective line $P^1(\mathbb{C})$, where the components of the 1-spinor ξ^A are regarded as the homogeneous coordinates of a point in $P^1(\mathbb{C})$. The fundamental theorem of algebra then guarantees four solutions (properly counted) of this equation, the five distinct modes of coincidence of which give the five Petrov types, whilst the actual solutions give the associated Debever–Penrose directions.

The spinor equivalent of the trace-free Ricci tensor also has four indices, but now two indices are undotted and two dotted and so the above approach must be modified if it is to be applied here. Penrose (1972) has shown how to carry out such a modification. The spinor $\phi_{AB\dot{X}\dot{Y}}$ determines the quartic equation

$$\phi_{AB\dot{X}\dot{Y}}\xi^A\xi^B\bar{\eta}^{\dot{X}}\bar{\eta}^{\dot{Y}} = 0 \qquad (\phi_{AB\dot{X}\dot{Y}} \neq 0) \tag{3.1}$$

the solutions of which determine certain points in $P^1(\mathbb{C}) \times P^1(\mathbb{C})$, where again the spinor components ξ^A and $\bar{\eta}^{\dot{X}}$ are regarded as homogeneous coordinates in the two copies of $P^1(\mathbb{C})$. Equation (3.1) may be viewed geometrically as follows: the 2-spinor $\xi^A\bar{\eta}^{\dot{X}}$ and its complex multiples determine a complex null direction. Hence it follows that (3.1) is equivalent to the two equations

$$\tilde{R}_{ab}x^a x^b = 0, \qquad g_{ab}x^a x^b = 0 \qquad (\tilde{R}_{ab} \neq 0) \tag{3.2}$$

where x is the complex (null) vector equivalent of $\xi^A \bar{\eta}^{\dot{X}}$. The equations in (3.2) determine two quadric surfaces in complex projective 3-space $P^3(\mathbb{C})$,[2] where the components x^a are homogeneous coordinates of a point in $P^3(\mathbb{C})$. Since the metric tensor is nondegenerate, the second quadric in (3.2) is proper and so one can choose coordinates X, Y, Z, T in $P^3(\mathbb{C})$ such that this quadric has equation $XT = YZ$. In this coordinate system, if $x^a \leftrightarrow \xi^A \bar{\eta}^{\dot{X}}$ then $(x^1, x^2, x^3, x^4) = (\xi^1\bar{\eta}^{\dot{1}}, \xi^1\bar{\eta}^{\dot{2}}, \xi^2\bar{\eta}^{\dot{1}}, \xi^2\bar{\eta}^{\dot{2}})$. Now the totality of complex null directions correspond in a one-to-one fashion to the totality of points on the second quadric in (3.2) (which shall be denoted by $\mathfrak{B}$), the two 1-spinors ξ^A and $\bar{\eta}^{\dot{X}}$ which are determined by a complex null direction up to a complex factor determining the point in $\mathfrak{B}$ $[\approx P^1(\mathbb{C}) \times P^1(\mathbb{C})]$. The real null directions determine a section $\mathfrak{R}$ of $\mathfrak{B}$ given by those points on $\mathfrak{B}$ for which the corresponding spinors ξ^A and $\bar{\eta}^{\dot{X}}$ may be chosen to satisfy $\bar{\xi}^{\dot{A}} = \bar{\eta}^{\dot{A}}$. This section is homeomorphic to $P^1(\mathbb{C})$ and hence to the 2-sphere S^2—the "sphere of vision." Now any proper quadric surface is generated by two families (reguli) of lines of $P^3(\mathbb{C})$. These lines lie entirely in the quadric surface and are the only lines of $P^3(\mathbb{C})$ to do so. In the above representation, the reguli on the quadric $g_{ab}x^a x^b = 0$ are the two families of lines given, respectively, by $\xi^A = \text{const}$ and $\bar{\eta}^{\dot{X}} = \text{const}$. No two members of the same reguli intersect, but any two members of different reguli intersect in a unique point. These reguli play an important rôle in the theory.

Returning now to the quadric surfaces (3.2), their intersection, as is well known, is a curve μ in $P^3(\mathbb{C})$ which intersects the "general plane" in $P^3(\mathbb{C})$ four times—a *quartic* curve. This curve is viewed as a quartic curve on $\mathfrak{B}$ and the classification of $\phi_{AB\dot{X}\dot{Y}}$ then proceeds by an analysis and classification of μ. The classification of μ proceeds in the first instance by an examination of the ways in which μ may decompose into irreducible curves of lower order. The possible modes of splitting of a curve defined by two quadric surfaces are well known (see for example Semple and Kneebone, 1952) and can be stated in terms of the numbers of intersections (properly counted) of the various components of μ with the general member of the two systems of generators of $\mathfrak{B}$. Being a quartic, μ will intersect the general member of each system of generators twice. Now a general curve on $\mathfrak{B}$ which intersects the general member of one family of

[2] A discussion of the properties of quadric surfaces can be found in Semple and Kneebone (1952).

generators p times and the other family of generators q times is called a (p,q) curve on $\mathfrak{B}$. Thus μ may decompose into certain irreducible parts with the proviso of course that the sum of the first integers and the sum of the second integers in the pairs (p,q) for the various members of the decomposition are each equal to 2. The types of irreducible curve one finds in such a decomposition are (2,2), (1,2), (2,1), (1,1), (1,0), and (0,1). The first is the irreducible quartic, the second and third are twisted cubics, the fourth is a (plane) conic [being the intersection of the quadric and a plane in $P^3(\mathbb{C})$] and the fifth and sixth are lines on the quadric and so are members of one or other of the two reguli on the quadric. One notes that curves of type (2,0) and (0,2) are not irreducible since they must decompose (according to the fundamental theorem of algebra) into two curves of type (1,0) and two curves of type (0,1), respectively. Fortunately, the fact that the two quadrics in (3.2) simultaneously have real coefficients imposes certain restrictions on the decomposition of μ. Firstly, if a (1,0) curve appears in the decomposition, then so must its conjugate (0,1) curve. Hence the twisted cubic is no longer a possibility in the present case because this would require an irreducible decomposition into curves of type (2,1) and (0,1) or curves of type (1,2) and (1,0). For similar reasons, an irreducible (1,1) curve may be a "real" conic, and so may occur with either another real conic or with two lines (generators), or else a complex conic, in which case it necessarily occurs with its irreducible conjugate conic. One is thus left with four types of irreducible component which are labeled by Penrose as follows:

Q—irreducible (2,2) quartic curve with real equation;
C—irreducible (1,1) conic with real equation;
B—pair of complex conjugate irreducible conics each of type (1,1);
X—pair of complex conjugate lines [types (1,0) and (0,1)].

The list of possible decompositions for μ is thus Q, B, CC, C^2, CX, XX, and X^2. The difference between CC and C^2 is that CC represents two distinct real conics whereas C^2 represents a repeated real conic. Similar comments apply to the cases XX and X^2.

The decomposition of the curve μ is reflected in the decomposition of the associated spinor $\phi_{AB\dot{X}\dot{Y}}$. One can in fact decompose $\phi_{AB\dot{X}\dot{Y}}$ generally as

$$\phi_{AB}{}^{\dot{X}\dot{Y}} = G_{(A}{}^{(\dot{X}}H_{B)}{}^{\dot{Y})} + \bar{G}_{(A}{}^{(\dot{X}}\bar{H}_{B)}{}^{\dot{Y})} \tag{3.3}$$

this being the spinor form of (2.1). In (3.3) G and H are arbitrary 2-spinors which are not Hermitian in general and $\bar{G}$ and $\bar{H}$ denote the corresponding conjugate spinors. In general, (3.3) will correspond to the irreducible quartic case Q for the curve μ. If in (3.3) H can be chosen to be the spinor conjugate to G (to within sign) and if G does not decompose into the product of two 1-spinors, then one obtains the case B. The case CC occurs when G and H may be chosen as nondecomposable, distinct, Hermitian spinors (and hence $G=\bar{G}$, $H=\bar{H}$) and the case C^2 when G and H may be chosen nondecomposable, Hermitian, and $G=\pm H$. The case CX arises if G and H can be selected Hermitian and with H (but not G) decomposing as $H_{A\dot{X}}=K_A\bar{K}_{\dot{X}}$ for some 1-spinor K_A. Finally, the case XX occurs when G and H can both be chosen to be Hermitian and to decompose as $G_{A\dot{X}}=J_A\bar{J}_{\dot{X}}$ and $H_{A\dot{X}}=K_A\bar{K}_{\dot{X}}$, where J_A and K_A are 1-spinors which are not proportional, the X^2 case occurring when J_A and K_A may be chosen to be proportional.

The classification of μ may be further refined along the following lines. Firstly one can subdivide the cases Q and C according to the number of connected one-dimensional pieces of this curve which lie on the reality section $\mathfrak{R}$, this number being employed as a subscript on the corresponding symbol. This leads to subtypes Q, Q_1, and Q_2 for curves of type Q and subtypes C and C_1 for curves of type C, where, following Penrose, the subscript zero has been dropped. Secondly one can subdivide the types of curve according to their real multiple point structure. The types of multiple point which can occur are now listed, together with the symbols used for them:

n—real node with two real branches (a double point);
i—isolated real node with conjugate imaginary branches (a double point);
nn—two real nodes;
ii—two isolated real nodes;
c—cusp (a double point with one branch and coincident tangents);
τ—tacnode (a double point with two branches, real or imaginary, and coincident tangents);
τ^∞—real curve of tacnodes (a repeated real curve);
t—triple point with one real and two conjugate imaginary tangents;
q—quadruple point with two repeated conjugate imaginary tangents.

It is a consequence of the quartic nature of μ that if it possesses a triple point, a quadruple point, or more than one double point then it is necessarily reducible (that is, it is not in the class Q). Reducibility will also be seen to follow for a curve admitting a tacnode. Conversely if μ is reducible it necessarily admits multiple points. However, these multiple points might not be real and so need not appear in the classification list.

The existence of a multiple point imposes restrictions on $\phi_{AB\dot{X}\dot{Y}}$. In fact a point on $\mathfrak{B}$ with associated spinors ξ_A and $\bar{\eta}_{\dot{X}}$ is a multiple point of μ

if and only if

$$\phi_{AB\dot{X}\dot{Y}}\xi^A\bar{\eta}^{\dot{X}} = \lambda\xi_B\bar{\eta}_{\dot{Y}} \qquad (\lambda \in \mathbb{R}) \tag{3.4}$$

For a real multiple point one has (3.4) holding with $\bar{\eta}_{\dot{Y}} = \bar{\xi}_{\dot{Y}}$ and $\lambda \in \mathbb{R}$. So a complex (real) multiple point corresponds to a complex (real) null Ricci eigendirection (cf. the end of Section 1). A real cusp or a real tacnode will occur if and only if the two tangents at the real multiple point are coincident. This occurs when (3.4) holds (with $\bar{\eta}_{\dot{Y}} = \bar{\xi}_{\dot{Y}}$) and when in the equation

$$\phi_{AB\dot{X}\dot{Y}}\xi^A\xi^B = \nu\bar{\xi}_{\dot{X}}\bar{\xi}_{\dot{Y}} \tag{3.5}$$

[which is equivalent to (3.4) when $\bar{\xi}_{\dot{Y}} = \bar{\eta}_{\dot{Y}}$], $2|\lambda| = |\nu| \neq 0$. For a real node (real isolated node) it is necessary and sufficient that (3.4) holds with $\bar{\eta}_{\dot{Y}} = \bar{\xi}_{\dot{Y}}$ and that in (3.5), $2|\lambda| < |\nu|$ ($2|\lambda| > |\nu|$). A real triple point occurs if and only if

$$\phi_{AB\dot{X}\dot{Y}}\xi^A\xi^B = 0 = \phi_{AB\dot{X}\dot{Y}}\xi^A\bar{\xi}^{\dot{X}}, \qquad \phi_{AB\dot{X}\dot{Y}}\xi^A \neq 0 \tag{3.6}$$

Finally, a real quadruple point occurs if and only if

$$\phi_{AB\dot{X}\dot{Y}}\xi^A = 0 \tag{3.7}$$

These results should be compared with the results (i)–(iv) at the end of Section 1 and the comments which followed them.

The possibilities for the real multiple point structure and reducibility of μ can now be gathered together by considering each of the possibilities for the decomposition of μ mentioned earlier and checking which type of real multiple point can occur in each case. Using the multiple point symbols as suffices in an obvious notation, one arrives at the following list: $Q_2, Q_1, Q, Q_1i, Q_1n, Qi, Q_1c, Bii, C_1C_1nn, C_1C_1, B, C_1C, CC, B\tau, C_1C_1\tau, C_1Xi, CXi, C_1Xt, XXii, C_1^2\tau^\infty, C^2, X^2q$.

Before discussing the connection between the two classifications summarized in this section and in the last one, it is remarked that Penrose's classification has several interesting topological features and also the advantage that it enables a diagram of specializations of the above types to be drawn up easily (although the resulting diagram is somewhat complicated).

4. THE CONNECTION BETWEEN THE CLASSIFICATIONS

The connection between the schemes of Ludwig and Scanlon and of Penrose can now be achieved in a straightforward fashion. The details of the connection are given in the table. It is remarked that the subclassification according to sign in the Ludwig–Scanlon scheme (indicated by a $\pm$ sign in types A and C and by the extra splitting achieved in type B according to the sign of $P_a q^a$ and implicitly contained but not explicitly mentioned in type D) is not relevant to achieve the connection considered here. A similar distinction of sign was considered by Penrose and can easily be incorporated into the table if necessary.

The details of the table can now be discussed. It is readily shown that if a proper real conic intersects the reality section, it does so in a single one-dimensional connected region of $\mathfrak{R}$. The first three rows of the table now follow. Corresponding to the Ludwig–Scanlon class B_1 a or b, one easily finds a Penrose curve of type XX. This necessitates two isolated real nodes corresponding to the two real double points, each with conjugate imaginary branches, at which the two conjugate line pairs meet. For the Ludwig–Scanlon type B_2 one easily finds a Penrose curve of type C_1X. Here, one has two possibilities. One occurs when the real multiple point defined by the X component of this curve lies on the proper conic C_1 and the other occurs when it does not. In the first case one has a real triple point with one real and two conjugate imaginary tangents whilst in the latter case one has a single real double point with conjugate imaginary branches. (The possibility of a further subdivision here according to the complex multiple point structure is ruled out owing to the fact that the planes in which the line pair X and the conic C_1 lie are necessarily distinct.) The corresponding Ludwig–Scanlon types are B_{2a} and $B_{2b,c}$. For the Ludwig–Scanlon class B_3 one has a corresponding Penrose curve of type C_1C_1. Remembering that the conic C_1 uniquely determines the plane in $P^3(\mathbb{C})$ in which it lies, it follows that there are only three possibilities, namely, when the two conics have none, one, or two common real points. In the first case one has the Penrose type C_1C_1 and the lack of real multiple points shows that the Ludwig–Scanlon type is $B_{3f,g}$. In the second case it can be shown that the unique real multiple point is a double point with two real branches and coincident tangents and so the Penrose type is $C_1C_1\tau$ and the corresponding Ludwig–Scanlon type is easily shown to be $B_{3d,e}$. In the third case the two real multiple points are nodes and the corresponding types are C_1C_1nn and $B_{3a,b,c}$. For the Ludwig–Scanlon type B_4 we have a curve of type CX and the pair of conjugate lines gives an isolated real node so that the type is CXi. This is the only possibility here since the planes in which the X and C components lie are distinct. It is easily shown that the types B_5 and CC correspond as do the types B_6 and C_1C.

For the Ludwig–Scanlon class C one has a curve of Penrose type B corresponding to two conjugate proper conics. Three possibilities can occur: when there are no real multiple points (B), when there is one real double point (which turns out to be a tacnode and so gives the type $B\tau$), and where there are two real double points (with conjugate imaginary branches and hence gives the type Bii). The corresponding Ludwig–Scanlon types are $C_{1\pm}$, $C_{2\pm}$, and $C_{3\pm}$.

Finally the correspondence between the Ludwig–Scanlon type D and the Penrose type Q can be discussed. Here the situation is a little more complicated since Ludwig and Scanlon divide their type D according to Segré type only, whereas Penrose's scheme produced a significant refinement of this subdivision. Consider first the type D_1 corresponding to Segré type $\{1,1,1,1\}$ with all eigenvalues distinct. The canonical form is

$$\tilde{R}_{ab}=\rho_1 x_a x_b+\rho_2 y_a y_b+\rho_3 z_a z_b-\rho_4 t_a t_b \tag{4.1}$$

where (x, y, z, t) constitute a pseudo-orthonormal tetrad ($x^a x_a=y^a y_a=z^a z_a=-t^a t_a=1$ and all other inner products zero) and where $\rho_1, \rho_2, \rho_3, \rho_4$ are distinct real numbers and satisfy $\rho_1+\rho_2+\rho_3+\rho_4=0$. By considering the spinor form of (4.1) in terms of the canonical spinor basis α_A and β_A (see Appendix) and by considering solutions of the equation

$$\phi_{AB\dot{X}\dot{Y}}\xi^A\xi^B\bar{\xi}^{\dot{X}}\bar{\xi}^{\dot{Y}}=0 \tag{4.2}$$

where either $\xi^A=\beta^A$ or $\xi^A=\alpha^A+(x+iy)\beta^A$, $x, y\in\mathbb{R}$, one finds after some calculation that either the curve μ fails to intersect the reality section $\mathfrak{R}$ (and this occurs if and only if $\rho_4 \gtrless \rho_1, \rho_2, \rho_3$) or else it intersects it in two distinct one-dimensional connected pieces. Thus type D_1 corresponds to the curve types Q and Q_2 according to the above eigenvalue inequality, there being no real multiple points in this case. In the type-D_2 case one has a trace-free Ricci tensor with two nonreal conjugate eigenvalues and all eigenvalues distinct (Segré type $\{z, \bar{z}, 1, 1\}$). Here a similar calculation reveals that μ always intersects $\mathfrak{R}$ in a single one-dimensional connected piece, and since there are no real multiple points, one has a curve of type Q_1. In the D_3 case one has a trace-free Ricci tensor of Segré type $\{2,1,1\}$, with all eigenvalues distinct, and a canonical form

$$\tilde{R}_{ab}=2\sigma_1 l_{(a}m_{b)}+\lambda l_a l_b+\sigma_2 e_a e_b+\sigma_3 f_a f_b \tag{4.3}$$

where (l, m, e, f) constitute a real null tetrad ($l^a m_a=e^a e_a=f^a f_a=1$ and all other inner products zero). In (4.3), σ_1, σ_2, and σ_3 are distinct real numbers, $\lambda \in \mathbb{R}$, and $2\sigma_1 + \sigma_2 + \sigma_3 = 0$. Also if $\lambda > 0 (< 0)$ then it may be set equal to 1 (-1) by a judicious choice of the null tetrad without affecting (4.3). In this case one finds that μ intersects $\mathfrak{R}$ in a single one-dimensional connected piece if and only if $\sigma_2 \gtrless \sigma_1 \gtrless \sigma_3$ (in which case a single real multiple point occurs which is a node and gives type $Q_1 n$) and that μ intersects $\mathfrak{R}$ in a single one-dimensional connected piece together with a single isolated point if and only if $\sigma_1 \lambda > 0$ and $\sigma_1 \gtrless \sigma_2, \sigma_3$ (in which case a single real multiple point occurs which is an isolated node and gives type $Q_1 i$). Otherwise μ intersects $\mathfrak{R}$ in a single point which is an isolated real node and gives the type Qi. In type D_4 one has the Segré type $\{3, 1\}$ with distinct eigenvalues and finds that μ intersects $\mathfrak{R}$ in a single one-dimensional connected piece and that a single real multiple point occurs which is a cusp, corresponding therefore to type $Q_1 c$.

APPENDIX

Here, the connections between the various representations of the trace-free Ricci tensor can be discussed. Let α_A and β_A constitute a spinor basis with $\alpha_A \beta^A = 1$ and construct a complex null tetrad $(l, m, t, \bar{t})$ according to $l_a \leftrightarrow \alpha_A \bar{\alpha}_{\dot{X}}$, $m_a \leftrightarrow -\beta_A \bar{\beta}_{\dot{X}}$, $t_a \leftrightarrow \beta_A \bar{\alpha}_{\dot{X}}$ $(\bar{t}_a \leftrightarrow \alpha_A \bar{\beta}_{\dot{X}})$. Thus $l^a m_a = t^a \bar{t}_a = 1$ with all other inner products zero. (Recall that the signature is taken to be $+2$ and so $g_{ab} \leftrightarrow -\varepsilon_{AB} \varepsilon_{\dot{W}\dot{X}}$.) One can construct from this null tetrad the usual complex self-dual bivectors,

$$V_{ab} = 2 l_{[a} \bar{t}_{b]} \leftrightarrow \varepsilon_{\dot{X}\dot{Y}} \alpha_A \alpha_B$$

$$U_{ab} = 2 m_{[a} t_{b]} \leftrightarrow \varepsilon_{\dot{X}\dot{Y}} \beta_A \beta_B \tag{A.1}$$

$$M_{ab} = 2 l_{[a} m_{b]} + 2 \bar{t}_{[a} t_{b]} \leftrightarrow -2 \varepsilon_{\dot{X}\dot{Y}} \alpha_{(A} \beta_{B)}$$

Finally one can decompose R_{ab}, $\tilde{R}_{ab}$, $\overset{+}{E}_{abcd}$, and $\phi_{AB\dot{X}\dot{Y}}$ in the forms below:

$$R_{ab} = 2R^1 l_{(a} m_{b)} + R^2 l_a l_b + R^3 m_a m_b + 2R^4 l_{(a} x_{b)} + 2R^5 l_{(a} y_{b)}$$
$$+ 2R^6 m_{(a} x_{b)} + 2R^7 m_{(a} y_{b)} + 2R^8 x_{(a} y_{b)} + R^9 x_a x_b + R^{10} y_a y_b. \tag{A.2}$$

where $R^1, R^2, \ldots, R^{10} \in \mathbb{R}$, where $2^{1/2}t_a = x_a + iy_a$ [so that (l, m, x, y) constitute a real null tetrad] and where a similar equation holds for $\tilde{R}_{ab}$ with a tilde on the corresponding coefficients.

$$\overset{+}{E}_{abcd} = E_1\bar{U}_{ab}U_{cd} + E_2\bar{V}_{ab}V_{cd} + E_3\bar{M}_{ab}M_{cd} + E_4\bar{U}_{ab}V_{cd} + E_5\bar{V}_{ab}U_{cd}$$
$$+ E_6\bar{U}_{ab}M_{cd} + E_7\bar{M}_{ab}U_{cd} + E_8\bar{V}_{ab}M_{cd} + E_9\bar{M}_{ab}V_{cd} \tag{A.3}$$

where $E_1, E_2, \ldots, E_9 \in \mathbb{C}$ with $E_1, E_2, E_3 \in \mathbb{R}$, $E_5 = \bar{E}_4$, $E_7 = \bar{E}_6$, $E_9 = \bar{E}_8$.

$$\phi_{AB\dot{X}\dot{Y}} = \phi_{00}\alpha_A\alpha_B\bar{\alpha}_{\dot{X}}\bar{\alpha}_{\dot{Y}} + 2\phi_{01}\alpha_A\alpha_B\bar{\alpha}_{(\dot{X}}\bar{\beta}_{\dot{Y})} + 2\bar{\phi}_{01}\alpha_{(A}\beta_{B)}\bar{\alpha}_{\dot{X}}\bar{\alpha}_{\dot{Y}}$$
$$+ 4\phi_{11}\alpha_{(A}\beta_{B)}\bar{\alpha}_{(\dot{X}}\bar{\beta}_{\dot{Y})} + \phi_{02}\alpha_A\alpha_B\bar{\beta}_{\dot{X}}\bar{\beta}_{\dot{Y}} + \bar{\phi}_{02}\beta_A\beta_B\bar{\alpha}_{\dot{X}}\bar{\alpha}_{\dot{Y}}$$
$$+ 2\phi_{12}\alpha_{(A}\beta_{B)}\bar{\beta}_{\dot{X}}\bar{\beta}_{\dot{Y}} + 2\bar{\phi}_{12}\beta_A\beta_B\bar{\alpha}_{(\dot{X}}\bar{\beta}_{\dot{Y})} + \phi_{22}\beta_A\beta_B\bar{\beta}_{\dot{X}}\bar{\beta}_{\dot{Y}} \tag{A.4}$$

The various coefficients here are connected as follows:

$$\tilde{R}^1 = \tfrac{1}{2}R^1 - \tfrac{1}{4}(R^9 + R^{10}), \qquad \tilde{R}^9 = -\tfrac{1}{2}R^1 + \tfrac{3}{4}R^9 - \tfrac{1}{4}R^{10}$$
$$\tilde{R}^{10} = -\tfrac{1}{2}R^1 - \tfrac{1}{4}R^9 + \tfrac{3}{4}R^{10}, \qquad \tilde{R}^i = R^i \qquad (i \neq 1, 9, 10) \tag{A.5}$$

$$2\phi_{00} = E_2, \qquad 2\phi_{01} = -\bar{E}_8, \qquad 2\phi_{11} = E_3$$
$$2\phi_{02} = E_4, \qquad 2\phi_{12} = -E_6, \qquad 2\phi_{22} = E_1 \tag{A.6}$$

$$E_1 = \tilde{R}^3, \qquad E_2 = \tilde{R}^2, \qquad E_3 = -\tilde{R}^1, \qquad E_4 = \tilde{R}^1 + \tilde{R}^9 + i\tilde{R}^8$$
$$E_6 = 2^{-1/2}(\tilde{R}^6 + i\tilde{R}^7), \qquad E_8 = -2^{-1/2}(\tilde{R}^4 - i\tilde{R}^5) \tag{A.7}$$

$\tilde{R}_{ab}$ must assume one of the four Segré types $\{1,1,1,1\}$, $\{2,1,1\}$, $\{3,1\}$, and $\{z, \bar{z}, 1, 1\}$ or their degeneracies. The canonical forms for all these types have been given (see for example Hall, 1976, 1979) and can be used, together with the above connecting formulas, to obtain canonical forms for any of the above representations of $\tilde{R}_{ab}$.

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